

Second-Order Necessary/Sufficient Conditions for Optimal Control Problems in the Absence of Linear Structure ^{*}

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Abstract. Second-order necessary conditions for optimal control problems are considered, where the “second-order” is in the sense of that Pontryagin’s maximum principle is viewed as a first-order necessary optimality condition. A sufficient condition for a local minimizer is also given.

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1. Introduction. We will give a new kind of second-order necessary/sufficient conditions for optimal controls. Second-order necessary/sufficient conditions for optimal control problems have been studied for a long time. There are many relevant works. Among them, we mention the following works and the references therein: [1]—[8], [11]—[15], [18]—[30] and [32]. There are different definitions of “first-order necessary/sufficient conditions” and “second-order necessary/sufficient conditions”. To our best knowledge, the corresponding first-order necessary conditions to these second-order conditions in the literature are not Pontryagin’s maximum principle. In addition, the control domains considered there are domains (or closed domains) in \mathbb{R}^m . In other words, second-order necessary/sufficient conditions for optimal controls in the literature are mainly used to distinguish optimal controls from other singular controls in the classical sense, not from other singular controls in the sense of Pontryagin’s maximum principle. For the definitions of singular controls in the classical sense and singular controls in the sense of Pontryagin’s maximum principle, see Definitions 1 and 2 in [11], see also (4.5) and (4.3).

Before we focus on our problems, we recall the results about necessary conditions for minimizers of functions.

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Let us consider a minimizer x_0 of a smooth function $f(\cdot)$ on $\overline{\Omega}$, where $\overline{\Omega}$ is the closure of a domain $\Omega \subseteq \mathbb{R}^n$. We call a unit vector ℓ is an admissible direction if there exists a $\delta > 0$ such that $x_0 + s\ell \in \overline{\Omega}$ for any $s \in [0, \delta]$. If ℓ is admissible, we have the following first-order necessary condition:

$$0 \leq \lim_{s \rightarrow 0^+} \frac{f(x_0 + s\ell) - f(x_0)}{s} = \langle \nabla f(x_0), \ell \rangle. \quad (1.1)$$

When

$$\langle \nabla f(x_0), \ell \rangle = 0 \quad (1.2)$$

holds, i.e., (1.1) degenerates, then we can get further the second-order necessary condition:

$$0 \leq \lim_{s \rightarrow 0^+} \frac{f(x_0 + s\ell) - f(x_0)}{s^2} = \frac{1}{2} \langle D^2 f(x_0) \ell, \ell \rangle, \quad (1.3)$$

where $D^2 f$ is the Hessian matrix of f . If (1.2) does not hold, that is

$$\langle \nabla f(x_0), \ell \rangle > 0,$$

then (1.3) does not necessarily hold.

From the above observations, we see that to yield second-order conditions of a minimizer, linear structure of independent variables is needed and second-order conditions only appear when first-order conditions degenerate.

For an optimal control problem, usually the control domain U need not have linear structure. Thus, the space \mathcal{U}_{ad} of control functions need not have linear structure. Pontryagin's maximum principle is a kind of necessary conditions that a minimizer satisfies. Many people look it as the first-order necessary condition. However, Pontryagin's maximum principle could not be obtained **directly** in a way like (1.2). First, for an optimal control $\bar{u}(\cdot)$, there is probably no "admissible direction" $v(\cdot)$ such that $\bar{u}(\cdot) + sv(\cdot)$ is still in \mathcal{U}_{ad} . Secondly, even if "admissible direction $v(\cdot)$ " exists, what we could get from

$$0 \leq \lim_{s \rightarrow 0^+} \frac{J(\bar{u}(\cdot) + sv(\cdot)) - J(\bar{u}(\cdot))}{s}$$

is only a corollary of Pontryagin's maximum principle which looks like (4.4), where we denote $J(\cdot)$ the cost functional of the optimal control problem.

When linear structure lacks, could we replace the "admissible direction" by "admissible path"? In other words, could we replace $\bar{u}(\cdot) + sv(\cdot)$ by $u_s(\cdot) \in \mathcal{U}_{ad}$, which is continuous in some sense in $s \in [0, 1]$? Certainly, we can do that. Yet, "admissible path" will immediately puzzles us on what are first-order conditions and second-order conditions. To see this, let us consider the function $f(\cdot)$ and its minimizer x_0 again. Let ℓ be an admissible direction such that (1.2) holds. Then choosing $x(s) = x_0 + \sqrt{s}\ell$, we have

$$0 \leq \lim_{s \rightarrow 0^+} \frac{f(x(s)) - f(x_0)}{s} = \frac{1}{2} \langle D^2 f(x_0) \ell, \ell \rangle. \quad (1.4)$$

Then, should we call (1.4) a first-order condition? Therefore, we think it is not a good idea to replace “admissible direction” by “admissible path”. In this paper, we will transform the original optimal control problem to a new problem, which is in fact the locally relaxed problem of the original problem. In this new problem, the corresponding space of control functions has linear structure and we can yield Pontryagin’s maximum principle like (1.1) under this linear structure. Then we can further yield second-order conditions based on Pontryagin’s maximum principle.

To reveal our idea clearly, we consider simply optimal control problems governed by ordinary differential equations.

The rest of the paper is organized as follows: In Section 2, we will give a method to linearize the control space near the optimal control. In Section 3, We will give a new proof of Pontryagin’s maximum principle. Section 4 will be devoted to second-order necessary conditions of optimality. Finally, a sufficient condition for a control being a local minimizer will be given in Section 5.

2. Local Linearization of Optimal Control Problems. In this section, we will linearize locally an optimal control problem along its minimizer. Let us consider the following controlled system:

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)), & \text{in } [0, T], \\ x(0) = x_0 \end{cases} \quad (2.1)$$

and the following cost functional

$$J(u(\cdot)) = \int_0^T f^0(t, x(t), u(t)) dt, \quad (2.2)$$

where $T > 0$, and $u(\cdot) \in \mathcal{U}_{ad}$ with

$$\mathcal{U}_{ad} = \{v : [0, T] \rightarrow U \mid v(\cdot) \text{ measurable}\}. \quad (2.3)$$

We pose the following assumptions:

(S1) The metric space (U, ρ) is separable.

(S2) Functions $\mathbf{f} = \begin{pmatrix} f^0 \\ f \end{pmatrix} = (f^0 \ f^1 \ f^2 \ \dots \ f^n)^\top : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^{n+1}$ are measurable in t , continuous in (x, u) and continuously differentiable in x , where B^\top denotes the transposition of a matrix B . Moreover, there exists a constant $L > 0$ such that

$$\begin{cases} |\mathbf{f}(t, x, u) - \mathbf{f}(t, \hat{x}, u)| \leq L|x - \hat{x}|, \\ |\mathbf{f}(t, 0, u)| \leq L, \end{cases} \quad \forall (t, x, \hat{x}, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times U. \quad (2.4)$$

Now, let $\bar{u}(\cdot) \in \mathcal{U}_{ad}$ be a minimizer of $J(\cdot)$ over \mathcal{U}_{ad} . We linearize \mathcal{U}_{ad} along $\bar{u}(\cdot)$ in the following manner. Define

$$\mathcal{M}_{ad} \equiv \{(1 - \alpha)\delta_{\bar{u}(\cdot)} + \alpha\delta_{u(\cdot)} \mid \alpha \in [0, 1], u(\cdot) \in \mathcal{U}_{ad}\}, \quad (2.5)$$

where δ_v denotes the Dirac measure at v on U . For an element $\sigma(\cdot) = (1 - \alpha)\delta_{\bar{u}(\cdot)} + \alpha\delta_{u(\cdot)} \in \mathcal{M}_{ad}$, denote

$$\mathbf{f}(t, x, \sigma(t)) \equiv \int_U \mathbf{f}(t, x, v)\sigma(t)(dv) = (1 - \alpha)\mathbf{f}(t, x, \bar{u}(t)) + \alpha\mathbf{f}(t, x, u(t)). \quad (2.6)$$

Then we can define $x(\cdot) = x(\cdot; \sigma(\cdot))$ as the solution of the equation

$$\begin{cases} \dot{x}(t) = f(t, x(t), \sigma(t)), & \text{in } [0, T], \\ x(0) = x_0 \end{cases} \quad (2.7)$$

and the corresponding cost functional $J(\sigma(\cdot))$ by

$$J(\sigma(\cdot)) \equiv \int_0^T f^0(t, x(t; \sigma(\cdot)), \sigma(t)) dt. \quad (2.8)$$

We can see that $x(\cdot; u(\cdot))$ and $J(u(\cdot))$ coincide with $x(\cdot; \delta_{u(\cdot)})$ and $J(\delta_{u(\cdot)})$ respectively. Thus, \mathcal{U}_{ad} can be viewed as a subset of \mathcal{M}_{ad} in the sense of identifying $u(\cdot) \in \mathcal{U}_{ad}$ to $\delta_{u(\cdot)} \in \mathcal{M}_{ad}$. Readers who are familiar with relaxed controls will immediately find \mathcal{M}_{ad} is a subset of relaxed control space. Yet, elements of \mathcal{M}_{ad} are much simpler than other relaxed controls. This is why we need neither to pose additional assumptions like that the control domain is compact as Warga did (c.f. [31]) nor to introduce the relaxed control defined by finite-additive probability measure as Fattorini did (c.f. [9]). \mathcal{M}_{ad} has a linear structure at $\bar{u}(\cdot)$, i.e., it contains all elements in the form $\delta_{\bar{u}(\cdot)} + \alpha(\delta_{u(\cdot)} - \delta_{\bar{u}(\cdot)})$ ($\alpha \in [0, 1]$). It can be proved easily that $\delta_{\bar{u}(\cdot)}$ is a minimizer of $J(\sigma(\cdot))$ over \mathcal{M}_{ad} . Using this fact, we can derive Pontryagin's maximum principle from

$$0 \leq \lim_{\alpha \rightarrow 0^+} \frac{J((1 - \alpha)\delta_{\bar{u}(\cdot)} + \alpha\delta_{u(\cdot)}) - J(\delta_{\bar{u}(\cdot)})}{\alpha}. \quad (2.9)$$

It is easy to prove the following results.

Lemma 2.1. *Let (S1)–(S2) hold. Then, there exists a constant $C > 0$, such that for any $\sigma(\cdot) \in \mathcal{M}_{ad}$,*

$$\begin{cases} \|x(\cdot; \sigma(\cdot))\|_{C[0, T]} \leq C, \\ |x(t; \sigma(\cdot)) - x(\hat{t}; \sigma(\cdot))| \leq C|t - \hat{t}|. \end{cases} \quad (2.10)$$

Proof. Let $\sigma(\cdot) \in \mathcal{M}_{ad}$ and $x(\cdot) = x(\cdot; \sigma(\cdot))$. We have

$$x(t) = x_0 + \int_0^t f(s, x(s), \sigma(s)) ds, \quad \forall t \in [0, T].$$

Then it follows from (S2) that

$$\begin{aligned} |x(t)| &\leq |x_0| + \left| \int_0^t f(s, 0, \sigma(s)) ds \right| + L \int_0^t |x(s)| ds \\ &\leq |x_0| + LT + L \int_0^t |x(s)| ds, \quad \forall t \in [0, T]. \end{aligned}$$

Thus, by Gronwall's inequality,

$$|x(t)| \leq (|x_0| + LT)e^{Lt}, \quad \forall t \in [0, T]. \quad (2.11)$$

Consequently, by (S2),

$$|f(t, x(t), \sigma(t))| \leq L + L|x(t)| \leq L + L(|x_0| + LT)e^{LT}, \quad \forall t \in [0, T].$$

Therefore

$$|x(t) - x(\hat{t})| = \left| \int_{\hat{t}}^t f(s, x(s), \sigma(s)) ds \right| \leq L \left(1 + (|x_0| + LT)e^{LT} \right) |t - \hat{t}|, \quad \forall t, \hat{t} \in [0, T]. \quad (2.12)$$

We get (2.10) from (2.11)–(2.12). \square

Lemma 2.2. *Let (S1)–(S2) hold and $\bar{u}(\cdot)$ be a minimizer of $J(u(\cdot))$ over \mathcal{U}_{ad} . Then $\delta_{\bar{u}(\cdot)}$ is a minimizer of $J(\sigma(\cdot))$ over \mathcal{M}_{ad} .*

Proof. Fix $\alpha \in [0, 1]$ and $u(\cdot) \in \mathcal{U}_{ad}$. Denote $\sigma^\alpha(\cdot) = (1 - \alpha)\delta_{\bar{u}(\cdot)} + \alpha\delta_{u(\cdot)}$ and $x^\alpha(\cdot) = x(\cdot; \sigma^\alpha(\cdot))$.

We will prove that

$$J(\delta_{\bar{u}(\cdot)}) \leq J(\sigma^\alpha(\cdot)). \quad (2.13)$$

For $\varepsilon > 0$, define

$$u^{\alpha, \varepsilon}(t) = \begin{cases} u(t), & \text{if } \{\frac{t}{\varepsilon}\} \in [0, \alpha), \\ \bar{u}(t), & \text{if } \{\frac{t}{\varepsilon}\} \in [\alpha, 1), \end{cases}$$

where $\{a\}$ denotes the decimal part of a real number a . Then $u^{\alpha, \varepsilon}(\cdot) \in \mathcal{U}_{ad}$. Denote $x^{\alpha, \varepsilon}(\cdot) = x(\cdot; u^{\alpha, \varepsilon}(\cdot))$. Then by Lemma 2.1, $x^{\alpha, \varepsilon}(\cdot)$ is uniformly bounded and equicontinuous on $[0, T]$. Consequently, by Arzelà-Ascoli's theorem, along a subsequence $\varepsilon \rightarrow 0^+$, $x^{\alpha, \varepsilon}(\cdot)$ converges uniformly to some $y(\cdot)$ in $[0, T]$. Thus, using a generalization of Riemann-Lebesgue's Theorem (see Ch. II, Theorem 4.15 in [33]), we can easily prove that by a subsequence $\varepsilon \rightarrow 0^+$,

$$f(t, y(t), u^{\alpha, \varepsilon}(t)) \rightarrow f(t, y(t), \sigma^\alpha(t)), \quad \text{weakly in } L^2(0, T; \mathbb{R}^n). \quad (2.14)$$

Since

$$|f(t, x^{\alpha, \varepsilon}(t), u^{\alpha, \varepsilon}(t)) - f(t, y(t), u^{\alpha, \varepsilon}(t))| \leq L |x^{\alpha, \varepsilon}(t) - y(t)|,$$

it follows from (2.14) that

$$f(t, x^{\alpha, \varepsilon}(t), u^{\alpha, \varepsilon}(t)) \rightarrow f(t, y(t), \sigma^\alpha(t)), \quad \text{weakly in } L^2(0, T; \mathbb{R}^n). \quad (2.15)$$

Similarly,

$$f^0(t, x^{\alpha, \varepsilon}(t), u^{\alpha, \varepsilon}(t)) \rightarrow f^0(t, y(t), \sigma^\alpha(t)), \quad \text{weakly in } L^2(0, T). \quad (2.16)$$

Passing to the limit for $\varepsilon \rightarrow 0^+$ in the following equality

$$x^{\alpha, \varepsilon}(t) = x_0 + \int_0^t f(s, x^{\alpha, \varepsilon}(s), u^{\alpha, \varepsilon}(s)) ds,$$

we get from (2.15) that

$$y(t) = x_0 + \int_0^t f(s, y(s), \sigma^\alpha(s)) ds,$$

i.e., $y(\cdot) = x^\alpha(\cdot)$. Furthermore, we can see that $x^{\alpha, \varepsilon}(\cdot)$ itself converges uniformly to $x^\alpha(\cdot)$ in $[0, T]$. Combining this with (2.16), we have

$$J(\delta_{\bar{u}(\cdot)}) \leq \lim_{\varepsilon \rightarrow 0^+} J(u^{\alpha, \varepsilon}(\cdot)) = J(\sigma^\alpha(\cdot)). \quad (2.17)$$

□

3. Pontryagin's Maximum Principle. Now, we will derive Pontryagin's maximum principle from (2.9). The idea of our proof could be tracked back to the works on relaxed control (c.f. [10] and [31], for example). However, one can still find that the proof we will give later has some improvement. Moreover, it can also be used to problems governed by partial differential equations and even having state constraints (c.f. [17]).

We keep the notations used in §2 and denote $\bar{x}(\cdot) = x(\cdot; \bar{u}(\cdot))$. We have

$$\begin{aligned} X^\alpha(t) &\equiv \frac{x^\alpha(t) - \bar{x}(t)}{\alpha} = \int_0^t \left[\frac{f(s, x^\alpha(s), \bar{u}(s)) - f(s, \bar{x}(s), \bar{u}(s))}{\alpha} \right. \\ &\quad \left. + f(s, x^\alpha(s), u(s)) - f(s, x^\alpha(s), \bar{u}(s)) \right] ds \\ &= \int_0^t \left[\int_0^1 f_x(s, \bar{x}(s) + \tau(x^\alpha(s) - \bar{x}(s)), \bar{u}(s))^\top d\tau X^\alpha(s) \right. \\ &\quad \left. + f(s, x^\alpha(s), u(s)) - f(s, x^\alpha(s), \bar{u}(s)) \right] ds, \end{aligned} \quad (3.1)$$

where $f_x(t, x, u)$ denotes the transpose of the Jacobi matrix of f on x . By (3.1), (S2), and using the same argument as the proof of the uniform convergence of $x^\alpha(\cdot) \rightarrow \bar{x}(\cdot)$ in $[0, T]$, we can easily get

$$X^\alpha(\cdot) \rightarrow X(\cdot), \quad \text{uniformly in } [0, T] \quad (3.2)$$

and $X(\cdot)$ solves the variational equation

$$\begin{cases} \dot{X}(t) = f_x(t, \bar{x}(t), \bar{u}(t))^\top X(t) + f(t, \bar{x}(t), u(t)) - f(t, \bar{x}(t), \bar{u}(t)), & \text{in } [0, T], \\ X(0) = 0. \end{cases} \quad (3.3)$$

Now, by introducing the adjoint equation

$$\begin{cases} \dot{\bar{\psi}}(t) = -f_x(t, \bar{x}(t), \bar{u}(t)) \bar{\psi}(t) + f_x^0(t, \bar{x}(t), \bar{u}(t)), & \text{in } [0, T], \\ \bar{\psi}(T) = 0, \end{cases} \quad (3.4)$$

we get from (2.17) and Lebesgue's dominated convergence theorem that

$$\begin{aligned}
0 &\leq \lim_{\alpha \rightarrow 0^+} \frac{J((1-\alpha)\delta_{\bar{u}(\cdot)} + \alpha\delta_{u(\cdot)}) - J(\delta_{\bar{u}(\cdot)})}{\alpha} \\
&= \lim_{\alpha \rightarrow 0^+} \int_0^T \left[\frac{f^0(t, x^\alpha(t), \bar{u}(t)) - f^0(t, \bar{x}(t), \bar{u}(t))}{\alpha} \right. \\
&\quad \left. + f^0(t, x^\alpha(t), u(t)) - f^0(t, x^\alpha(t), \bar{u}(t)) \right] dt \\
&= \lim_{\alpha \rightarrow 0^+} \int_0^T \left[\int_0^1 \langle f_x^0(t, \bar{x}(t) + s(x^\alpha(t) - \bar{x}(t)), \bar{u}(t)), X^\alpha(t) \rangle ds \right. \\
&\quad \left. + f^0(t, x^\alpha(t), u(t)) - f^0(t, x^\alpha(t), \bar{u}(t)) \right] dt \\
&= \int_0^T \left[\langle f_x^0(t, \bar{x}(t), \bar{u}(t)), X(t) \rangle + f^0(t, \bar{x}(t), u(t)) - f^0(t, \bar{x}(t), \bar{u}(t)) \right] dt \\
&= \int_0^T \left[H(t, \bar{x}(t), \bar{u}(t), \bar{\psi}(t)) - H(t, \bar{x}(t), u(t), \bar{\psi}(t)) \right] dt, \tag{3.5}
\end{aligned}$$

where

$$H(t, x, u, \psi) \equiv \langle f(t, x, u), \psi \rangle - f^0(t, x, u), \quad \forall (t, x, u, \psi) \in [0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n. \tag{3.6}$$

Then, since U is separable and H is continuous in u , it follows from (3.5) and a standard argument that

$$H(t, \bar{x}(t), \bar{u}(t), \bar{\psi}(t)) = \max_{v \in U} H(t, \bar{x}(t), v, \bar{\psi}(t)), \quad \text{a.e. } t \in [0, T]. \tag{3.7}$$

Relations (3.4), (3.6), (3.7) form Pontryagin's maximum principle.

4. Second-Order Necessary Optimality Conditions. We turn to study second-order necessary optimality conditions where Pontryagin's maximum principle is viewed as a first-order necessary optimality condition. In other words, we will give a second-order necessary condition for optimality to distinguish singular controls in the sense of Pontryagin's maximum principle. One can see that in (3.7), the equality holds if and only if

$$u(t) \in \bar{U}(t) \equiv \left\{ w \mid H(t, \bar{x}(t), w, \bar{\psi}(t)) = \max_{v \in U} H(t, \bar{x}(t), v, \bar{\psi}(t)) \right\}, \quad \text{a.e. } t \in [0, T]. \tag{4.1}$$

In this case,

$$\int_0^T \left[\langle f_x^0(t, \bar{x}(t), \bar{u}(t)), X(t) \rangle + f^0(t, \bar{x}(t), u(t)) - f^0(t, \bar{x}(t), \bar{u}(t)) \right] dt = 0. \tag{4.2}$$

Denote

$$\bar{U}_{ad} = \{ v(\cdot) \in \mathcal{U}_{ad} \mid v(t) \in \bar{U}(t), \quad \text{a.e. } t \in [0, T] \}. \tag{4.3}$$

Elements in \bar{U}_{ad} are called singular controls in the sense of Pontryagin's maximum principle. If U is an open subset of \mathbb{R}^m , then

$$H_u(t, \bar{x}(t), \bar{u}(t), \bar{\psi}(t)) = 0, \quad \text{a.e. } t \in [0, T]. \tag{4.4}$$

In this case, we call elements in

$$\{v(\cdot) \in \mathcal{U}_{ad} \mid H_u(t, \bar{x}(t), v(t), \bar{\psi}(t)) = 0, \quad \text{a.e. } t \in [0, T]\} \quad (4.5)$$

as singular controls in the classical sense (see Definitions 1 and 2 in [11]).

Now we make the following assumption:

(S3) Functions \mathbf{f} are twice continuously differentiable in x . Moreover, it holds that

$$|\mathbf{f}_x(t, x, u) - \mathbf{f}_x(t, \hat{x}, u)| \leq L|x - \hat{x}|, \quad \forall (t, x, \hat{x}, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times U. \quad (4.6)$$

We mention that (S2) implies

$$|\mathbf{f}_x(t, x, u)| \leq L, \quad \forall (t, x, u) \in [0, T] \times \mathbb{R}^n \times U$$

and (S3) implies

$$|f_{xx}^k(t, x, u)| \leq L, \quad \forall (t, x, u) \in [0, T] \times \mathbb{R}^n \times U$$

for $k = 0, 1, 2, \dots, n$.

The following theorem gives second-order necessary optimality conditions.

Theorem 4.1. *Let (S1)—(S3) hold and $\bar{u}(\cdot)$ be a minimizer of $J(\cdot)$ over \mathcal{U}_{ad} . Define $\bar{W}(\cdot)$ be the solution of the following second-order adjoint equation:*

$$\begin{cases} \dot{\bar{W}}(t) + f_x(t, \bar{x}(t), \bar{u}(t))\bar{W}(t) + \bar{W}(t)f_x(t, \bar{x}(t), \bar{u}(t))^\top \\ \quad + H_{xx}(t, \bar{x}(t), \bar{u}(t), \bar{\psi}(t)) = 0, \quad \text{in } [0, T], \\ \bar{W}(T) = 0 \end{cases} \quad (4.7)$$

and $\bar{\Phi}(\cdot)$ be the solution of

$$\begin{cases} \dot{\bar{\Phi}}(t) = f_x(t, \bar{x}(t), \bar{u}(t))^\top \bar{\Phi}(t), \quad \text{in } [0, T], \\ \bar{\Phi}(0) = I, \end{cases} \quad (4.8)$$

where I is the unit $n \times n$ matrix. Then for any $u(\cdot) \in \bar{\mathcal{U}}_{ad}$,

$$\begin{aligned} & \int_0^T dt \int_0^t \left\langle \bar{W}(t)(f(t, \bar{x}(t), \bar{u}(t)) - f(t, \bar{x}(t), u(t))) \right. \\ & \quad \left. + H_x(t, \bar{x}(t), \bar{u}(t), \bar{\psi}(t)) - H_x(t, \bar{x}(t), u(t), \bar{\psi}(t)), \right. \\ & \quad \left. \bar{\Phi}(t)\bar{\Phi}(s)^{-1}(f(s, \bar{x}(s), \bar{u}(s)) - f(s, \bar{x}(s), u(s))) \right\rangle ds \leq 0. \end{aligned} \quad (4.9)$$

Proof. Let $u(\cdot) \in \bar{\mathcal{U}}_{ad}$. Then by (4.2), for any $\alpha \in (0, 1]$,

$$\begin{aligned} & J((1 - \alpha)\delta_{\bar{u}(\cdot)} + \alpha\delta_{u(\cdot)}) - J(\delta_{\bar{u}(\cdot)}) \\ &= \alpha \int_0^T \left[\int_0^1 \langle f_x^0(t, \bar{x}(t) + s(x^\alpha(t) - \bar{x}(t)), \bar{u}(t)), X^\alpha(t) \rangle ds \right. \end{aligned}$$

$$\begin{aligned}
& + f^0(t, x^\alpha(t), u(t)) - f^0(t, x^\alpha(t), \bar{u}(t)) \Big] dt \\
= & \alpha \int_0^T dt \int_0^1 \langle f_x^0(t, \bar{x}(t) + s(x^\alpha(t) - \bar{x}(t)), \bar{u}(t)) - f_x^0(t, \bar{x}(t), \bar{u}(t)), X^\alpha(t) \rangle ds \\
& + \alpha \int_0^T \langle f_x^0(t, \bar{x}(t), \bar{u}(t)), X^\alpha(t) - X(t) \rangle dt \\
& + \alpha \int_0^T \left[(f^0(t, x^\alpha(t), u(t)) - f^0(t, \bar{x}(t), u(t))) - (f^0(t, x^\alpha(t), \bar{u}(t)) - f^0(t, \bar{x}(t), \bar{u}(t))) \right] dt \\
= & \alpha^2 \int_0^T dt \int_0^1 ds \int_0^1 s \langle f_{xx}^0(t, \bar{x}(t) + s\tau(x^\alpha(t) - \bar{x}(t)), \bar{u}(t)) X^\alpha(t), X^\alpha(t) \rangle d\tau \\
& + \alpha^2 \int_0^T \langle f_x^0(t, \bar{x}(t), \bar{u}(t)), Y^\alpha(t) \rangle dt \\
& + \alpha^2 \int_0^T dt \int_0^1 \langle f_x^0(t, \bar{x}(t) + s(x^\alpha(t) - \bar{x}(t)), u(t)), X^\alpha(t) \rangle ds \\
& - \alpha^2 \int_0^T dt \int_0^1 \langle f_x^0(t, \bar{x}(t) + s(x^\alpha(t) - \bar{x}(t)), \bar{u}(t)), X^\alpha(t) \rangle ds, \tag{4.10}
\end{aligned}$$

where $Y^\alpha(\cdot) \equiv (Y_1^\alpha(\cdot) \ Y_2^\alpha(\cdot) \ \dots \ Y_n^\alpha(\cdot))^\top = \frac{X^\alpha(\cdot) - X(\cdot)}{\alpha}$ satisfies

$$\begin{aligned}
Y_k^\alpha(t) &= \frac{1}{\alpha} \int_0^t \left[\int_0^1 \langle f_x^k(s, \bar{x}(s) + \tau(x^\alpha(s) - \bar{x}(s)), \bar{u}(s)), X^\alpha(s) \rangle d\tau \right. \\
&\quad \left. + f^k(s, x^\alpha(s), u(s)) - f^k(s, x^\alpha(s), \bar{u}(s)) \right] ds \\
&\quad - \frac{1}{\alpha} \int_0^t \left[\langle f_x^k(s, \bar{x}(s), \bar{u}(s)), X(s) \rangle + f^k(s, \bar{x}, u(s)) - f^k(s, \bar{x}(s), \bar{u}(s)) \right] ds \\
&= \int_0^t ds \int_0^1 d\tau \int_0^1 \tau \langle f_{xx}^k(s, \bar{x}(s) + \tau\zeta(x^\alpha(s) - \bar{x}(s)), \bar{u}(s)) X^\alpha(s), X^\alpha(s) \rangle d\zeta \\
&\quad + \int_0^t ds \int_0^1 \langle f_x^k(s, \bar{x}(s) + \tau(x^\alpha(s) - \bar{x}(s)), u(s)), X^\alpha(s) \rangle d\tau \\
&\quad - \int_0^t ds \int_0^1 \langle f_x^k(s, \bar{x}(s) + \tau(x^\alpha(s) - \bar{x}(s)), \bar{u}(s)), X^\alpha(s) \rangle d\tau \\
&\quad + \int_0^t \langle f_x^k(s, \bar{x}(s), \bar{u}(s)), Y^\alpha(s) \rangle ds, \quad k = 1, 2, \dots, n. \tag{4.11}
\end{aligned}$$

Using (S3) and by the same way to derive (3.2), we can get

$$Y^\alpha(\cdot) \rightarrow Y(\cdot), \quad \text{uniformly in } [0, T] \tag{4.12}$$

where $Y(\cdot)$ solves the following second variational equation

$$\left\{ \begin{array}{l} \dot{Y}(t) = f_x(t, \bar{x}(t), \bar{u}(t))^\top Y(t) + (f_x(t, \bar{x}(t), u(t)) - f_x(t, \bar{x}(t), \bar{u}(t)))^\top X(t) \\ \quad + \frac{1}{2} \begin{pmatrix} \langle f_{xx}^1(t, \bar{x}(t), \bar{u}(t))X(t), X(t) \rangle \\ \langle f_{xx}^2(t, \bar{x}(t), \bar{u}(t))X(t), X(t) \rangle \\ \vdots \\ \langle f_{xx}^n(t, \bar{x}(t), \bar{u}(t))X(t), X(t) \rangle \end{pmatrix}, \quad \text{in } [0, T], \\ Y(0) = 0. \end{array} \right. \quad (4.13)$$

Then it follows from (4.10) and Lebesgue's dominated convergence theorem that

$$\begin{aligned} 0 &\leq \lim_{\alpha \rightarrow 0^+} \frac{J((1-\alpha)\delta_{\bar{u}(\cdot)} + \alpha\delta_{u(\cdot)}) - J(\delta_{\bar{u}(\cdot)})}{\alpha^2} \\ &= \int_0^T \langle f_x^0(t, \bar{x}(t), \bar{u}(t)), Y(t) \rangle dt + \int_0^T \langle f_x^0(t, \bar{x}(t), u(t)) - f_x^0(t, \bar{x}(t), \bar{u}(t)), X(t) \rangle dt \\ &\quad + \frac{1}{2} \int_0^T \langle f_{xx}^0(t, \bar{x}(t), \bar{u}(t))X(t), X(t) \rangle dt \\ &= \int_0^T \langle H_x(t, \bar{x}(t), \bar{u}(t), \bar{\psi}(t)) - H_x(t, \bar{x}(t), u(t), \bar{\psi}(t)), X(t) \rangle dt \\ &\quad - \frac{1}{2} \int_0^T \langle H_{xx}(t, \bar{x}(t), \bar{u}(t), \bar{\psi}(t))X(t), X(t) \rangle dt \\ &= \int_0^T \langle H_x(t, \bar{x}(t), \bar{u}(t), \bar{\psi}(t)) - H_x(t, \bar{x}(t), u(t), \bar{\psi}(t)), X(t) \rangle dt \\ &\quad - \frac{1}{2} \text{tr} \left[\int_0^T H_{xx}(t, \bar{x}(t), \bar{u}(t), \bar{\psi}(t))X(t)X(t)^\top dt \right], \end{aligned} \quad (4.14)$$

where $\text{tr } B$ denotes the trace of a matrix B . One can easily verify that

$$\left\{ \begin{array}{l} \frac{d}{dt}(X(t)X(t)^\top) = f_x(t, \bar{x}(t), \bar{u}(t))^\top (X(t)X(t)^\top) + (X(t)X(t)^\top) f_x(t, \bar{x}(t), \bar{u}(t)) \\ \quad + (f_x(t, \bar{x}(t), u(t)) - f_x(t, \bar{x}(t), \bar{u}(t)))X(t)^\top \\ \quad + X(t)(f_x(t, \bar{x}(t), u(t)) - f_x(t, \bar{x}(t), \bar{u}(t)))^\top, \quad \text{in } [0, T], \\ X(0)X(0)^\top = 0. \end{array} \right. \quad (4.15)$$

Now, we introduce the second-order adjoint equation (4.7). By (S1)—(S3), we can see that (4.7) admits a unique solution $\bar{W}(\cdot)$. Since $\bar{W}(\cdot)^\top$ satisfies (4.7) too, $\bar{W}(\cdot)$ should be symmetric. Since $\text{tr}(AB) = \text{tr}(BA)$ for all $k \times j$ matrix A and $j \times k$ matrix B , we have

$$\begin{aligned} & -\frac{1}{2} \text{tr} \left[\int_0^T H_{xx}(t, \bar{x}(t), \bar{u}(t), \bar{\psi}(t))X(t)X(t)^\top dt \right] \\ &= \frac{1}{2} \text{tr} \left[\int_0^T \left(\frac{d}{dt} \bar{W}(t) + f_x(t, \bar{x}(t), \bar{u}(t))\bar{W}(t) + \bar{W}(t)f_x(t, \bar{x}(t), \bar{u}(t))^\top \right) X(t)X(t)^\top dt \right] \\ &= \frac{1}{2} \text{tr} \left\{ \int_0^T \left[-\bar{W}(t) \frac{d}{dt}(X(t)X(t)^\top) + f_x(t, \bar{x}(t), \bar{u}(t))\bar{W}(t)X(t)X(t)^\top \right. \right. \end{aligned}$$

$$\begin{aligned}
& + \overline{W}(t) f_x(t, \bar{x}(t), \bar{u}(t))^\top X(t) X(t)^\top \Big] dt \Big\} \\
& = \frac{1}{2} \operatorname{tr} \left\{ \int_0^T \left[-\overline{W}(t) f_x(t, \bar{x}(t), \bar{u}(t))^\top X(t) X(t)^\top - \overline{W}(t) X(t) X(t)^\top f_x(t, \bar{x}(t), \bar{u}(t)) \right. \right. \\
& \quad - \overline{W}(t) (f(t, \bar{x}(t), u(t)) - f(t, \bar{x}(t), \bar{u}(t))) X(t)^\top \\
& \quad - \overline{W}(t) X(t) (f(t, \bar{x}(t), u(t)) - f(t, \bar{x}(t), \bar{u}(t)))^\top \\
& \quad \left. \left. + f_x(t, \bar{x}(t), \bar{u}(t)) \overline{W}(t) X(t) X(t)^\top + \overline{W}(t) f_x(t, \bar{x}(t), \bar{u}(t))^\top X(t) X(t)^\top \right] dt \right\} \\
& = \int_0^T \langle \overline{W}(t) (f(t, \bar{x}(t), \bar{u}(t)) - f(t, \bar{x}(t), u(t))), X(t) \rangle dt. \tag{4.16}
\end{aligned}$$

By (3.3) and (4.7),

$$X(t) = \int_0^t \overline{\Phi}(t) \overline{\Phi}(s)^{-1} (f(s, \bar{x}(s), u(s)) - f(s, \bar{x}(s), \bar{u}(s))) ds.$$

Thus, it follows from (4.14) and (4.16) that

$$\begin{aligned}
0 & \leq \int_0^T \left\langle \overline{W}(t) (f(t, \bar{x}(t), \bar{u}(t)) - f(t, \bar{x}(t), u(t))) \right. \\
& \quad \left. + H_x(t, \bar{x}(t), \bar{u}(t), \bar{\psi}(t)) - H_x(t, \bar{x}(t), u(t), \bar{\psi}(t)), X(t) \right\rangle dt \\
& = \int_0^T dt \int_0^t \left\langle \overline{W}(t) (f(t, \bar{x}(t), \bar{u}(t)) - f(t, \bar{x}(t), u(t))) + H_x(t, \bar{x}(t), \bar{u}(t), \bar{\psi}(t)) \right. \\
& \quad \left. - H_x(t, \bar{x}(t), u(t), \bar{\psi}(t)), \overline{\Phi}(t) \overline{\Phi}(s)^{-1} (f(s, \bar{x}(s), u(s)) - f(s, \bar{x}(s), \bar{u}(s))) \right\rangle ds.
\end{aligned}$$

Therefore, we finish the proof. \square

To get an analogue of the maximum condition in Pontryagin's maximum principle, we introduce a lemma concerned with the well-known Filippov's Lemma or measurable selection. We recall that a Polish space is a separable completely metrizable topological space. We mention that all (nonempty) closed sets and open sets in \mathbb{R}^m are Polish spaces.

Lemma 4.2. *Let X be a Polish space, $\mathcal{T} \subseteq \mathbb{R}^n$ be a Lebesgue measurable set. Assume that $\Gamma : \mathcal{T} \rightarrow 2^X$ is measurable (i.e., for any closed set F , $\{t \in \mathcal{T} | \Gamma(t) \in F\}$ is measurable) and takes values on the family of nonempty closed subsets of X . Then $\Gamma(\cdot)$ admits a measurable selection, i.e., there exists a Lebesgue measurable map $\gamma : \mathcal{T} \rightarrow X$, such that*

$$\gamma(t) \in \Gamma(t), \quad \text{a.e. } t \in \mathcal{T}.$$

Lemma 4.2 and its proof can be found in [16] (Ch. 3, Theorem 2.23). Based on this lemma, we have that

Theorem 4.3. *Let U be a Polish space. Then under assumptions of Theorem 4.1, for almost all $t \in [0, T]$, it holds that:*

$$\begin{aligned} & \left\langle \overline{W}(t)(f(t, \bar{x}(t), \bar{u}(t)) - f(t, \bar{x}(t), v)) + H_x(t, \bar{x}(t), \bar{u}(t), \bar{\psi}(t)) \right. \\ & \left. - H_x(t, \bar{x}(t), v, \bar{\psi}(t)), (f(t, \bar{x}(t), \bar{u}(t)) - f(t, \bar{x}(t), v)) \right\rangle \leq 0, \quad \forall v \in \overline{U}(t). \end{aligned} \quad (4.17)$$

Proof. Denote

$$\begin{aligned} F(t, u) &= \overline{\Phi}(t)^\top \left[\overline{W}(t)(f(t, \bar{x}(t), \bar{u}(t)) - f(t, \bar{x}(t), u)) \right. \\ & \quad \left. + H_x(t, \bar{x}(t), \bar{u}(t), \bar{\psi}(t)) - H_x(t, \bar{x}(t), u, \bar{\psi}(t)) \right], \\ G(t, u) &= \overline{\Phi}(t)^{-1}(f(t, \bar{x}(t), u) - f(t, \bar{x}(t), \bar{u}(t))), \\ & \quad (t, u) \in [0, T] \times U. \end{aligned}$$

Then by Theorem 4.1, for any $u(\cdot) \in \overline{U}_{ad}$, we have

$$\int_0^T dt \int_0^t \langle F(t, u(t)), G(s, u(s)) \rangle ds \leq 0.$$

Let

$$E_{u(\cdot)} = \left\{ t \in [0, T] \mid \lim_{\alpha \rightarrow 0^+} \frac{1}{\alpha} \int_t^{t+\alpha} \begin{pmatrix} F(s, u(s)) \\ |F(s, u(s))|^2 \\ G(s, u(s)) \end{pmatrix} ds = \begin{pmatrix} F(t, u(t)) \\ |F(t, u(t))|^2 \\ G(t, u(t)) \end{pmatrix} \right\}. \quad (4.18)$$

Then $E_{u(\cdot)}$ has Lebesgue measure T . Let $\beta \in E_{u(\cdot)}$. For $\alpha \in (0, T - \beta)$, define

$$u^\alpha(t) = \begin{cases} \bar{u}(t), & \text{if } t \notin [\beta, \beta + \alpha], \\ u(t), & \text{if } t \in [\beta, \beta + \alpha]. \end{cases}$$

Then, $u^\alpha(\cdot) \in \overline{U}_{ad}$ and

$$\int_\beta^{\beta+\alpha} dt \int_\beta^t \langle F(t, u(t)), G(s, u(s)) \rangle ds = \int_0^T dt \int_0^t \langle F(t, u^\alpha(t)), G(s, u^\alpha(s)) \rangle ds \leq 0. \quad (4.19)$$

It is easy to see that under assumptions (S1)–(S3), $F(\cdot, \cdot)$, $G(\cdot, \cdot)$ are uniformly bounded. Thus

$$|F(t, u)| + |G(t, u)| \leq C, \quad \forall t \in [0, T] \times U$$

for some constant $C > 0$. Consequently,

$$\begin{aligned} & \left| \frac{2}{\alpha^2} \int_\beta^{\beta+\alpha} dt \int_\beta^t \langle F(t, u(t)), G(s, u(s)) \rangle ds - \langle F(\beta, u(\beta)), G(\beta, u(\beta)) \rangle \right| \\ & \leq \left| \frac{2}{\alpha^2} \int_\beta^{\beta+\alpha} \left\langle (t - \beta)F(t, u(t)), \frac{1}{t - \beta} \int_\beta^t G(s, u(s)) ds - G(\beta, u(\beta)) \right\rangle dt \right| \\ & \quad + \left| \left\langle \frac{2}{\alpha^2} \int_\beta^{\beta+\alpha} (t - \beta)(F(t, u(t)) - F(\beta, u(\beta))) dt, G(\beta, u(\beta)) \right\rangle \right| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{2C}{\alpha^2} \int_{\beta}^{\beta+\alpha} (t-\beta) dt \sup_{r \in (\beta, \beta+\alpha]} \left| \frac{1}{r-\beta} \int_{\beta}^r G(s, u(s)) ds - G(\beta, u(\beta)) \right| \\
&\quad + 2C \left[\frac{1}{\alpha^3} \int_{\beta}^{\beta+\alpha} (t-\beta)^2 dt \right]^{\frac{1}{2}} \left[\frac{1}{\alpha} \int_{\beta}^{\beta+\alpha} |F(t, u(t)) - F(\beta, u(\beta))|^2 dt \right]^{\frac{1}{2}} \\
&= C \sup_{r \in [\beta, \beta+\alpha]} \left| \frac{1}{r-\beta} \int_{\beta}^r G(s, u(s)) ds - G(\beta, u(\beta)) \right| \\
&\quad + \frac{2\sqrt{3}C}{3} \left\{ \frac{1}{\alpha} \int_{\beta}^{\beta+\alpha} \left[|F(t, u(t))|^2 + |F(\beta, u(\beta))|^2 - 2 \langle F(t, u(t)), F(\beta, u(\beta)) \rangle \right] dt \right\}^{\frac{1}{2}}
\end{aligned}$$

Let $\alpha \rightarrow 0^+$, we get from (4.18) that

$$\lim_{\alpha \rightarrow 0^+} \frac{2}{\alpha^2} \int_{\beta}^{\beta+\alpha} dt \int_{\beta}^t \langle F(t, u(t)), G(s, u(s)) \rangle ds = \langle F(\beta, u(\beta)), G(\beta, u(\beta)) \rangle.$$

Therefore, combining the above with (4.19), we get

$$\langle F(\beta, u(\beta)), G(\beta, u(\beta)) \rangle \leq 0. \quad (4.20)$$

That is, for any $u(\cdot) \in \overline{U}_{ad}$,

$$\langle F(t, u(t)), G(t, u(t)) \rangle \leq 0, \quad \text{a.e. } t \in [0, T]. \quad (4.21)$$

For $k = 1, 2, \dots$, denote

$$\mathcal{T}_k = \left\{ t \in [0, T] \mid \exists v \in \overline{U}(t), \text{ s.t. } \langle F(t, v), G(t, v) \rangle \geq \frac{1}{k} \right\}.$$

Then, \mathcal{T}_k is measurable. We claim \mathcal{T}_k has zero measure. Otherwise, for any $t \in \mathcal{T}_k$,

$$\Gamma_k(t) \equiv \left\{ v \in \overline{U} \mid \langle F(t, v), G(t, v) \rangle \geq \frac{1}{k} \right\}$$

is a nonempty closed subset of U . It is easy to see that $\Gamma(\cdot)$ is measurable since $\langle F(t, v), G(t, v) \rangle$ is measurable in t and continuous in v . Thus, by Lemma 4.2, there exists a measurable function $\tilde{u}_k : \mathcal{T}_k \rightarrow \Gamma_k$. Define

$$u_k(t) = \begin{cases} \bar{u}(t), & \text{if } t \notin \mathcal{T}_k, \\ \tilde{u}_k(t), & \text{if } t \in \mathcal{T}_k. \end{cases}$$

Then $u_k(\cdot) \in \overline{U}_{ad}$ and

$$\langle F(t, u_k(t)), G(t, u_k(t)) \rangle \geq \frac{1}{k}, \quad \text{a.e. } t \in \mathcal{T}_k.$$

Contradict to (4.21). Therefore \mathcal{T}_k has zero measure. Consequently,

$$\mathcal{T} \equiv \left\{ t \in [0, T] \mid \exists v \in \overline{U}(t), \text{ s.t. } \langle F(t, v), G(t, v) \rangle > 0 \right\} = \bigcup_{k=1}^{\infty} \mathcal{T}_k$$

has zero measure too. That is, for almost all $t \in [0, T]$,

$$\langle F(t, v), G(t, v) \rangle \leq 0, \quad \forall v \in \overline{U}(t). \quad (4.22)$$

This completes the proof. \square

It is not necessary to suppose that U is a Polish space in yielding (4.21). However, usually, we can not get (4.22) from (4.21) if we only suppose that U is a separable metric space. To see this, we introduce the following example.

Example 1. Let $T > 0$ and $U \subset [0, T]$ be a non-measurable set which contains 0 and has no any subset of positive measure. Let

$$H(t, u) = -u^2(t - u)^2, \quad g(t, u) = u^2, \quad (t, u) \in [0, T] \times U.$$

Then H and g are smooth. Define

$$\overline{U}(t) \equiv \left\{ u \in U \mid H(t, u) = \max_{v \in U} H(t, v) \right\}, \quad t \in [0, T].$$

Then

$$\overline{U}(t) = \begin{cases} \{0\} & \text{if } t \notin U, \\ \{0, t\}, & \text{if } t \in U. \end{cases}$$

Thus, if

$$u(\cdot) \in \overline{U}_{ad} \equiv \{v : [0, T] \rightarrow U \mid v(\cdot) \text{ measurable}, v(t) \in \overline{U}(t), \text{ a.e. } [0, T]\},$$

we must have $u(t) = 0$, a.e. $t \in [0, T]$. Otherwise, if $E \equiv \{t \mid u(t) \neq 0\}$ has positive measure, then $E \subseteq U$. This contradicts to the assumption that U has no any subset of positive measure

Therefore, for any $u(\cdot) \in \overline{U}_{ad}$,

$$g(t, u(t)) \leq 0, \quad \text{a.e. } t \in [0, T]. \quad (4.23)$$

However,

$$\left\{ t \in [0, T] \mid \max_{v \in \overline{U}(t)} g(t, v) > 0 \right\} = U \setminus \{0\}$$

has not zero measure. This means that the following statement does not hold: for almost all $t \in [0, T]$,

$$g(t, v) \leq 0, \quad \forall v \in \overline{U}(t).$$

5. Sufficient Conditions. Now, we give a sufficient condition for a control being a local minimizer. In addition to (S1)—(S3), we suppose that

(S4) There exists a modulus of continuity $\omega : [0, +\infty) \rightarrow [0, +\infty)$, such that

$$\begin{cases} |\mathbf{f}(t, x, u) - \mathbf{f}(t, x, \hat{u})| \leq \omega(\rho(u, \hat{u})), \\ |\mathbf{f}_x(t, x, u) - \mathbf{f}_x(t, x, \hat{u})| \leq \omega(\rho(u, \hat{u})), \end{cases} \quad \forall (t, x, \hat{x}, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times U \quad (5.1)$$

and for $k = 0, 1, 2, \dots, n$, it holds that

$$\begin{aligned} |f_{xx}^k(t, x, u) - f_{xx}^k(t, \hat{x}, \hat{u})| &\leq \omega(|x - \hat{x}| + \rho(u, \hat{u})), \\ \forall (t, x, \hat{x}, u, \hat{u}) &\in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times U \times U. \end{aligned} \quad (5.2)$$

We note that (S2)—(S4) imply

$$\begin{cases} |\mathbf{f}(t, x, u) - \mathbf{f}(t, \hat{x}, \hat{u})| \leq L|x - \hat{x}| + \omega(\rho(u, \hat{u})), \\ |\mathbf{f}_x(t, x, u) - \mathbf{f}_x(t, \hat{x}, \hat{u})| \leq L|x - \hat{x}| + \omega(\rho(u, \hat{u})). \end{cases}$$

For $\bar{u}(\cdot), u(\cdot) \in \mathcal{U}_{ad}$ and $\alpha \in (0, 1]$, we keep the notations used in Sections 3 and 4. For notation simplicity, denote

$$\Theta(t) = \omega(\rho(u(t), \bar{u}(t))), \quad t \in [0, T] \quad (5.3)$$

and $x^\alpha(\cdot) = \bar{x}(\cdot)$, $X^\alpha(\cdot) = X(\cdot)$, $Y^\alpha(\cdot) = Y(\cdot)$ when $\alpha = 0$.

The following lemma gives estimates of $X^\alpha(\cdot)$ and $Y^\alpha(\cdot)$.

Lemma 5.1. *Assume (S1)—(S4), $\bar{u}(\cdot), u(\cdot) \in \mathcal{U}_{ad}$ and $\alpha \in [0, 1]$. Then for any $\alpha \in [0, 1]$,*

$$|X^\alpha(t)| \leq C \int_0^t \Theta(s) ds, \quad \forall t \in [0, T], \quad (5.4)$$

$$|X^\alpha(t)|^2 \leq C \int_0^t [\Theta(s)]^2 ds, \quad \forall t \in [0, T], \quad (5.5)$$

$$|Y^\alpha(t)| \leq C \int_0^t [\Theta(s)]^2 ds, \quad \forall t \in [0, T] \quad (5.6)$$

and

$$\begin{aligned} &|Y^\alpha(t) - Y(t)| \\ &\leq C \left[\omega \left(C \int_0^T \Theta(s) ds \right) + \int_0^T \Theta(s) ds \right] \int_0^t [\Theta(s)]^2 ds, \quad \forall t \in [0, T], \end{aligned} \quad (5.7)$$

where and hereafter, $C > 0$ denotes a constant, which is independent of $u(\cdot)$, $\alpha \in [0, 1]$, and may be different in different lines.

Proof. Let $\alpha \in [0, 1]$. By (S2)—(S4), (3.1) and (3.3),

$$|X^\alpha(t)| \leq \int_0^t \left[L|X^\alpha(s)| + \Theta(s) \right] ds, \quad \forall t \in [0, T]. \quad (5.8)$$

Then, (5.4) follows from Gronwall's inequality. While (5.5) follows from (5.4) and Cauchy-Schwarz's inequality.

Similarly, by (S2)—(S4), (4.11) and (4.13),

$$|Y^\alpha(t)| \leq C \int_0^t \left[|Y^\alpha(s)| + |X^\alpha(s)|^2 + \Theta(s) |X^\alpha(s)| \right] ds, \quad \forall t \in [0, T]. \quad (5.9)$$

Then using Gronwall's inequality again, we have

$$\begin{aligned} |Y^\alpha(t)| &\leq C \int_0^t \left[|X^\alpha(s)|^2 + \Theta(s) |X^\alpha(s)| \right] ds \\ &\leq C \int_0^t ds \int_0^s \left\{ [\Theta(\tau)]^2 + \Theta(s)\Theta(\tau) \right\} d\tau \\ &= C \int_0^t (t-s+1) [\Theta(s)]^2 ds \\ &\leq C \int_0^t [\Theta(s)]^2 ds, \quad \forall t \in [0, T]. \end{aligned} \quad (5.10)$$

That is, (5.6) holds.

The proof of (5.7) is similar but a little complex. Consider (4.11) and (4.13). We have

$$\begin{aligned} &\left| \int_0^t ds \int_0^1 d\tau \int_0^1 \tau \langle f_{xx}^k(s, \bar{x}(s) + \tau \zeta(x^\alpha(s) - \bar{x}(s)), \bar{u}(s)) X^\alpha(s), X^\alpha(s) \rangle d\zeta \right. \\ &\quad \left. - \frac{1}{2} \int_0^t \langle f_{xx}^k(s, \bar{x}(s), \bar{u}(s)) X(s), X(s) \rangle ds \right| \\ &= \left| \int_0^t ds \int_0^1 d\tau \int_0^1 \tau \langle f_{xx}^k(s, \bar{x}(s) + \tau \zeta(x^\alpha(s) - \bar{x}(s)), \bar{u}(s)) X^\alpha(s), X^\alpha(s) \rangle d\zeta \right. \\ &\quad - \int_0^t ds \int_0^1 d\tau \int_0^1 \tau \langle f_{xx}^k(s, \bar{x}(s), \bar{u}(s)) X^\alpha(s), X^\alpha(s) \rangle d\zeta \\ &\quad + \int_0^t ds \int_0^1 d\tau \int_0^1 \tau \langle f_{xx}^k(s, \bar{x}(s), \bar{u}(s)) (X^\alpha(s) + X(s)), \alpha Y^\alpha(s) \rangle d\zeta \left| \right. \\ &\leq \int_0^t ds \int_0^1 d\tau \int_0^1 \tau \omega(\tau \zeta \alpha |X^\alpha(s)|) |X^\alpha(s)|^2 d\zeta \\ &\quad + C \int_0^t ds \int_0^1 d\tau \int_0^1 \tau \alpha (|X^\alpha(s)| + |X(s)|) |Y^\alpha(s)| d\zeta \\ &\leq \int_0^t \omega(|X^\alpha(s)|) |X^\alpha(s)|^2 ds + C \int_0^t (|X^\alpha(s)| + |X(s)|) |Y^\alpha(s)| ds \\ &\leq C \int_0^t \left[\omega \left(C \int_0^s \Theta(\tau) d\tau \right) + \int_0^s \Theta(\tau) d\tau \right] \left\{ \int_0^s [\Theta(\tau)]^2 d\tau \right\} ds \\ &\leq C \left[\omega \left(C \int_0^T \Theta(\tau) d\tau \right) + \int_0^T \Theta(\tau) d\tau \right] \int_0^t [\Theta(s)]^2 ds. \end{aligned} \quad (5.11)$$

Similarly,

$$\begin{aligned} &\left| \int_0^t ds \int_0^1 \langle f_x^k(s, \bar{x}(s) + \tau(x^\alpha(s) - \bar{x}(s)), u(s)), X^\alpha(s) \rangle d\tau \right. \\ &\quad - \int_0^t ds \int_0^1 \langle f_x^k(s, \bar{x}(s) + \tau(x^\alpha(s) - \bar{x}(s)), \bar{u}(s)), X^\alpha(s) \rangle d\tau \\ &\quad \left. - \int_0^t \langle f_x^k(s, \bar{x}(s), u(s)), X(s) \rangle ds + \int_0^t \langle f_x^k(s, \bar{x}(s), \bar{u}(s)), X(s) \rangle ds \right| \end{aligned}$$

$$\begin{aligned}
&= \left| \int_0^t ds \int_0^1 d\tau \int_0^1 \langle \alpha \tau f_{xx}^k(s, \bar{x}(s) + \tau \zeta(x^\alpha(s) - \bar{x}(s)), u(s)) X^\alpha(s), X^\alpha(s) \rangle d\tau \right. \\
&\quad - \int_0^t ds \int_0^1 d\tau \int_0^1 \langle \alpha \tau f_{xx}^k(s, \bar{x}(s) + \tau \zeta(x^\alpha(s) - \bar{x}(s)), \bar{u}(s)) X^\alpha(s), X^\alpha(s) \rangle d\tau \\
&\quad \left. + \int_0^t \langle f_x^k(s, \bar{x}(s), u(s)), \alpha Y^\alpha(s) \rangle ds - \int_0^t \langle f_x^k(s, \bar{x}(s), \bar{u}(s)), \alpha Y^\alpha(s) \rangle ds \right| \\
&\leq \int_0^t \omega(|X^\alpha(s)|) |X^\alpha(s)|^2 ds + \int_0^t \Theta(s) |Y^\alpha(s)| ds \\
&\leq C \int_0^t \left[\omega \left(C \int_0^s \Theta(\tau) d\tau \right) + \Theta(s) \right] \left\{ \int_0^s [\Theta(\tau)]^2 d\tau \right\} ds \\
&\leq C \left[\omega \left(C \int_0^T \Theta(\tau) d\tau \right) + \int_0^T \Theta(\tau) d\tau \right] \int_0^t [\Theta(s)]^2 ds. \tag{5.12}
\end{aligned}$$

While

$$\begin{aligned}
&\left| \int_0^t \langle f_x^k(s, \bar{x}(s), \bar{u}(s)), Y^\alpha(s) \rangle ds - \int_0^t \langle f_x^k(s, \bar{x}(s), \bar{u}(s)), Y(s) \rangle ds \right| \\
&\leq \int_0^t L |Y^\alpha(s) - Y(s)| ds. \tag{5.13}
\end{aligned}$$

Combining (5.11)–(5.13) with (4.11) and (4.13), we get

$$\begin{aligned}
|Y^\alpha(t) - Y(t)| &\leq C \int_0^t |Y^\alpha(s) - Y(s)| ds \\
&\quad + C \left[\omega \left(C \int_0^T \Theta(\tau) d\tau \right) + \int_0^T \Theta(\tau) d\tau \right] \int_0^t [\Theta(s)]^2 ds. \tag{5.14}
\end{aligned}$$

Then Gronwall's inequality implies (5.7). \square

Now, we give a sufficient optimality condition in the following:

Theorem 5.2. Assume (S1)–(S4) hold and $\bar{u}(\cdot) \in \mathcal{U}_{ad}$ satisfy (3.4), (3.6), (3.7). Let $\omega(\cdot)$ be the function appeared in (S4). If there exists a $\beta > 0$, such that for any $u(\cdot) \in \mathcal{U}_{ad}$,

$$\begin{aligned}
&\int_0^T dt \int_0^t \left\langle \bar{W}(t)(f(t, \bar{x}(t), \bar{u}(t)) - f(t, \bar{x}(t), u(t))) \right. \\
&\quad + H_x(t, \bar{x}(t), \bar{u}(t), \bar{\psi}(t)) - H_x(t, \bar{x}(t), u(t), \bar{\psi}(t)), \\
&\quad \left. \bar{\Phi}(t) \bar{\Phi}(s)^{-1} (f(s, \bar{x}(s), \bar{u}(s)) - f(s, \bar{x}(s), u(s))) \right\rangle ds \\
&\leq -\beta \int_0^T \left[\omega(\rho(u(t), \bar{u}(t))) \right]^2 dt, \tag{5.15}
\end{aligned}$$

then there exists an $\varepsilon_0 > 0$, such that for any

$$\begin{aligned}
u(\cdot) \in \mathcal{V} &\equiv \left\{ v(\cdot) \in \mathcal{U}_{ad} \mid \int_0^T \omega(\rho(v(t), \bar{u}(t))) dt \leq \varepsilon_0 \right\}, \\
J(u(\cdot)) - J(\bar{u}(\cdot)) &\geq \frac{\beta}{2} \int_0^T \left[\omega(\rho(u(t), \bar{u}(t))) \right]^2 dt. \tag{5.16}
\end{aligned}$$

In particular, $\bar{u}(\cdot)$ minimizes $J(\cdot)$ over \mathcal{V} .

Proof. The proof of the theorem is very similar to that of (5.7) in Lemma 5.1. By (3.7),

$$\begin{aligned} & \int_0^T \left[\langle f_x^0(t, \bar{x}(t), \bar{u}(t)), X(t) \rangle + f^0(t, \bar{x}(t), u(t)) - f^0(t, \bar{x}(t), \bar{u}(t)) \right] dt \\ &= \int_0^T \left[H(t, \bar{x}(t), \bar{u}(t), \bar{\psi}(t)) - H(t, \bar{x}(t), u(t), \bar{\psi}(t)) \right] dt \geq 0. \end{aligned} \quad (5.17)$$

Then

$$\begin{aligned} & J((1-\alpha)\delta_{\bar{u}(\cdot)} + \alpha\delta_{u(\cdot)}) - J(\delta_{\bar{u}(\cdot)}) \\ &= \int_0^T \left[f^0(t, x^\alpha(t), \bar{u}(t)) - f^0(t, \bar{x}(t), \bar{u}(t)) + \alpha(f^0(t, x^\alpha(t), u(t)) - f^0(t, x^\alpha(t), \bar{u}(t))) \right] dt \\ &= \alpha \int_0^T \left[\int_0^1 \langle f_x^0(t, \bar{x}(t) + s(x^\alpha(t) - \bar{x}(t)), \bar{u}(t)), X^\alpha(t) \rangle ds \right. \\ & \quad \left. + f^0(t, x^\alpha(t), u(t)) - f^0(t, x^\alpha(t), \bar{u}(t)) \right] dt \\ &\geq \alpha \int_0^T dt \int_0^1 \langle f_x^0(t, \bar{x}(t) + s(x^\alpha(t) - \bar{x}(t)), \bar{u}(t)) - f_x^0(t, \bar{x}(t), \bar{u}(t)), X^\alpha(t) \rangle ds \\ & \quad + \alpha \int_0^T \langle f_x^0(t, \bar{x}(t), \bar{u}(t)), X^\alpha(t) - X(t) \rangle dt \\ & \quad + \alpha \int_0^T (f^0(t, x^\alpha(t), u(t)) - f^0(t, \bar{x}(t), u(t))) dt \\ & \quad - \alpha \int_0^T (f^0(t, x^\alpha(t), \bar{u}(t)) - f^0(t, \bar{x}(t), \bar{u}(t))) dt \\ &= \alpha^2 \int_0^T dt \int_0^1 ds \int_0^1 s \langle f_{xx}^0(t, \bar{x}(t) + s\tau(x^\alpha(t) - \bar{x}(t)), \bar{u}(t)) X^\alpha(t), X^\alpha(t) \rangle d\tau \\ & \quad + \alpha^2 \int_0^T \langle f_x^0(t, \bar{x}(t), \bar{u}(t)), Y^\alpha(t) \rangle dt \\ & \quad + \alpha^2 \int_0^T dt \int_0^1 \langle f_x^0(t, \bar{x}(t) + s(x^\alpha(t) - \bar{x}(t)), u(t)), X^\alpha(t) \rangle ds \\ & \quad - \alpha^2 \int_0^T dt \int_0^1 \langle f_x^0(t, \bar{x}(t) + s(x^\alpha(t) - \bar{x}(t)), \bar{u}(t)), X^\alpha(t) \rangle ds, \end{aligned} \quad (5.18)$$

By (S2)—(S4) and (5.4)—(5.6),

$$\begin{aligned} & \int_0^T dt \int_0^1 ds \int_0^1 s \langle f_{xx}^0(t, \bar{x}(t) + s\tau(x^\alpha(t) - \bar{x}(t)), \bar{u}(t)) X^\alpha(t), X^\alpha(t) \rangle d\tau \\ & \quad - \frac{1}{2} \int_0^T \langle f_{xx}^0(t, \bar{x}(t), \bar{u}(t)) X(t), X(t) \rangle dt \\ &= \int_0^T dt \int_0^1 ds \int_0^1 s \left\langle \left(f_{xx}^0(t, \bar{x}(t) + s\tau(x^\alpha(t) - \bar{x}(t)), \bar{u}(t)) - f_{xx}^0(t, \bar{x}(t), \bar{u}(t)) \right) X^\alpha(t), X^\alpha(t) \right\rangle d\tau \\ & \quad + \alpha \int_0^T dt \int_0^1 ds \int_0^1 s \langle f_{xx}^0(t, \bar{x}(t), \bar{u}(t)) (X^\alpha(t) + X(t)), Y^\alpha(t) \rangle d\tau \\ &\geq -C \int_0^T dt \int_0^1 ds \int_0^1 s \left[\omega(s\tau\alpha|X^\alpha(t)|) |X^\alpha(t)|^2 + (|X^\alpha(t)| + |X(t)|) |Y^\alpha(t)| \right] d\tau \\ &\geq -C \int_0^T \left[\omega\left(C \int_0^t \Theta(s) ds\right) + \int_0^t \Theta(s) ds \right] \left\{ \int_0^t [\Theta(s)]^2 ds \right\} dt \end{aligned}$$

$$\geq -C \left[\omega \left(C \int_0^T \Theta(s) ds \right) + \int_0^T \Theta(s) ds \right] \left\{ \int_0^T [\Theta(s)]^2 ds \right\}. \quad (5.19)$$

By (S2)—(S4) and (5.7),

$$\begin{aligned} & \int_0^T \langle f_x^0(t, \bar{x}(t), \bar{u}(t)), Y^\alpha(t) \rangle dt - \int_0^T \langle f_x^0(t, \bar{x}(t), \bar{u}(t)), Y(t) \rangle dt \\ & \geq -C \int_0^T |Y^\alpha(t) - Y(t)| dt \\ & \geq -C \left[\omega \left(C \int_0^T \Theta(s) ds \right) + \int_0^T \Theta(s) ds \right] \int_0^T [\Theta(s)]^2 ds. \end{aligned} \quad (5.20)$$

On the other hand, it follows from (S2)—(S4) and (5.6)—(5.7) that

$$\begin{aligned} & \int_0^T dt \int_0^1 \langle f_x^0(t, \bar{x}(t) + s(x^\alpha(t) - \bar{x}(t)), u(t)), X^\alpha(t) \rangle ds \\ & - \int_0^T dt \int_0^1 \langle f_x^0(t, \bar{x}(t) + s(x^\alpha(t) - \bar{x}(t)), \bar{u}(t)), X^\alpha(t) \rangle ds \\ & - \int_0^T \langle f_x^0(t, \bar{x}(t), u(t)), X(t) \rangle dt + \int_0^T \langle f_x^0(t, \bar{x}(t), \bar{u}(t)), X(t) \rangle dt \\ & = \int_0^T dt \int_0^1 ds \int_0^1 s\alpha \langle f_{xx}^0(t, \bar{x}(t) + s\tau(x^\alpha(t) - \bar{x}(t)), u(t)) X^\alpha(t), X^\alpha(t) \rangle d\tau \\ & - \int_0^T dt \int_0^1 ds \int_0^1 s\alpha \langle f_{xx}^0(t, \bar{x}(t) + s\tau(x^\alpha(t) - \bar{x}(t)), \bar{u}(t)) X^\alpha(t), X^\alpha(t) \rangle d\tau \\ & + \int_0^T dt \int_0^1 \langle f_x^0(t, \bar{x}(t), u(t)) - f_x^0(t, \bar{x}(t), \bar{u}(t)), \alpha Y^\alpha(t) \rangle ds \\ & \geq -C \int_0^T \Theta(t) [|X^\alpha(t)|^2 + |Y^\alpha(t)|^2] dt \\ & \geq -C \int_0^T \Theta(t) \left\{ \int_0^t [\Theta(s)]^2 ds \right\} dt \\ & \geq -C \int_0^T \Theta(t) dt \int_0^T [\Theta(s)]^2 ds. \end{aligned} \quad (5.21)$$

Therefore, combining (5.18)—(5.21) with (5.14), (4.14) and (4.16), we have

$$\begin{aligned} & \frac{J((1-\alpha)\delta_{\bar{u}(\cdot)} + \alpha\delta_{u(\cdot)}) - J(\delta_{\bar{u}(\cdot)})}{\alpha^2} \\ & \geq \int_0^T \langle f_x^0(t, \bar{x}(t), \bar{u}(t)), Y(t) \rangle dt + \int_0^T \langle f_x^0(t, \bar{x}(t), u(t)) - f_x^0(t, \bar{x}(t), \bar{u}(t)), X(t) \rangle dt \\ & + \frac{1}{2} \int_0^T \langle f_{xx}^0(t, \bar{x}(t), \bar{u}(t)) X(t), X(t) \rangle dt \\ & - C \left[\omega \left(C \int_0^T \Theta(t) dt \right) + \int_0^T \Theta(t) dt \right] \int_0^T [\Theta(t)]^2 ds \\ & = \int_0^T dt \int_0^t \left\langle \bar{W}(t)(f(t, \bar{x}(t), \bar{u}(t)) - f(t, \bar{x}(t), u(t))) + H_x(t, \bar{x}(t), \bar{u}(t), \bar{\psi}(t)) \right. \\ & \quad \left. - H_x(t, \bar{x}(t), u(t), \bar{\psi}(t)), \bar{\Phi}(t) \bar{\Phi}(s)^{-1} (f(s, \bar{x}(s), u(s)) - f(s, \bar{x}(s), \bar{u}(s))) \right\rangle ds \\ & - C \left[\omega \left(C \int_0^T \Theta(t) dt \right) + \int_0^T \Theta(t) dt \right] \int_0^T [\Theta(t)]^2 dt \end{aligned}$$

$$\geq \left[\beta - C \omega \left(C \int_0^T \Theta(t) dt \right) - C \int_0^T \Theta(t) dt \right] \int_0^T [\Theta(t)]^2 dt. \quad (5.22)$$

Since

$$\lim_{\varepsilon \rightarrow 0^+} C [\omega(C\varepsilon) + \varepsilon] = 0,$$

there exists an $\varepsilon_0 > 0$, independent of $\alpha \in (0, 1]$ and $u(\cdot)$, such that when

$$\int_0^T \Theta(t) dt \leq \varepsilon_0,$$

it holds that

$$\frac{J((1-\alpha)\delta_{\bar{u}(\cdot)} + \alpha\delta_{u(\cdot)}) - J(\delta_{\bar{u}(\cdot)})}{\alpha^2} \geq \frac{\beta}{2} \int_0^T [\Theta(t)]^2 dt. \quad (5.23)$$

Choosing $\alpha = 1$ in (5.23), we get (5.16) and finish the proof. \square

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